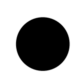
Lecture 30

Trees

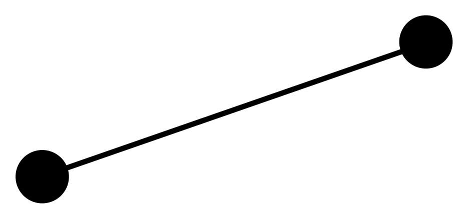
Trees

Definition: A **tree** is a connected graph with no cycles.

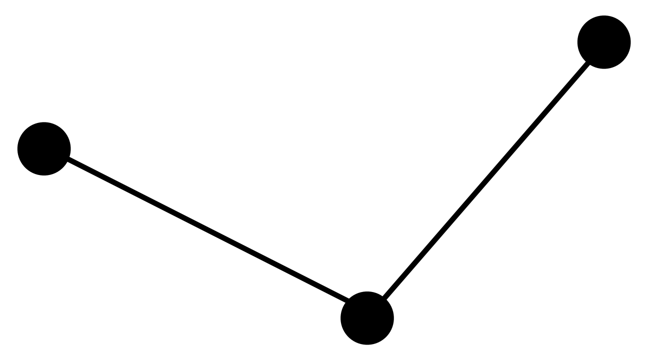
Examples:



$n = 1$

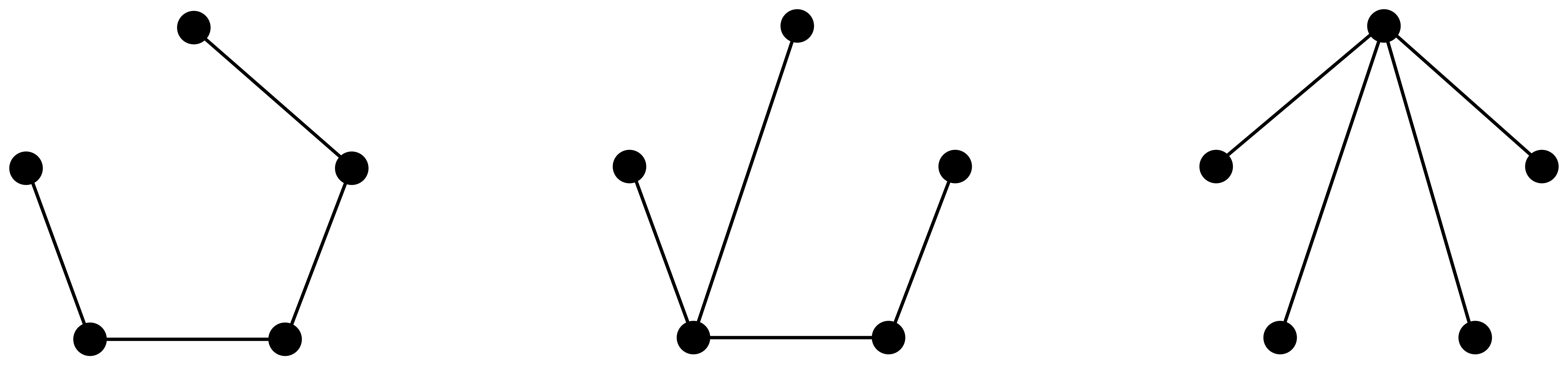


$n = 2$



$n = 3$

$n = 4$



$n = 5$

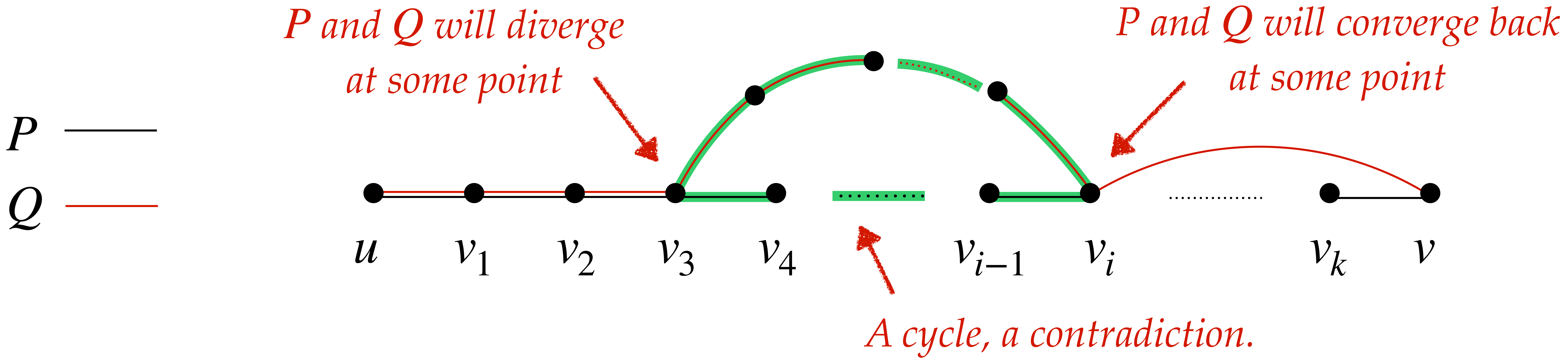
Alternative Characterisation of Trees

Theorem: The following assertions are equivalent for a graph G:

1) G is a tree.
2) Any two vertices are connected by a unique path in G.
3) G is minimally connected, i.e., G is connected but $G - e$ is disconnected for every edge $e \in G$.

Proof: We will prove 1) $\implies$ 2) $\implies$ 3) $\implies$ 1).

1) $\implies$ 2): Suppose there are vertices u and v in G connected with two paths P and Q.



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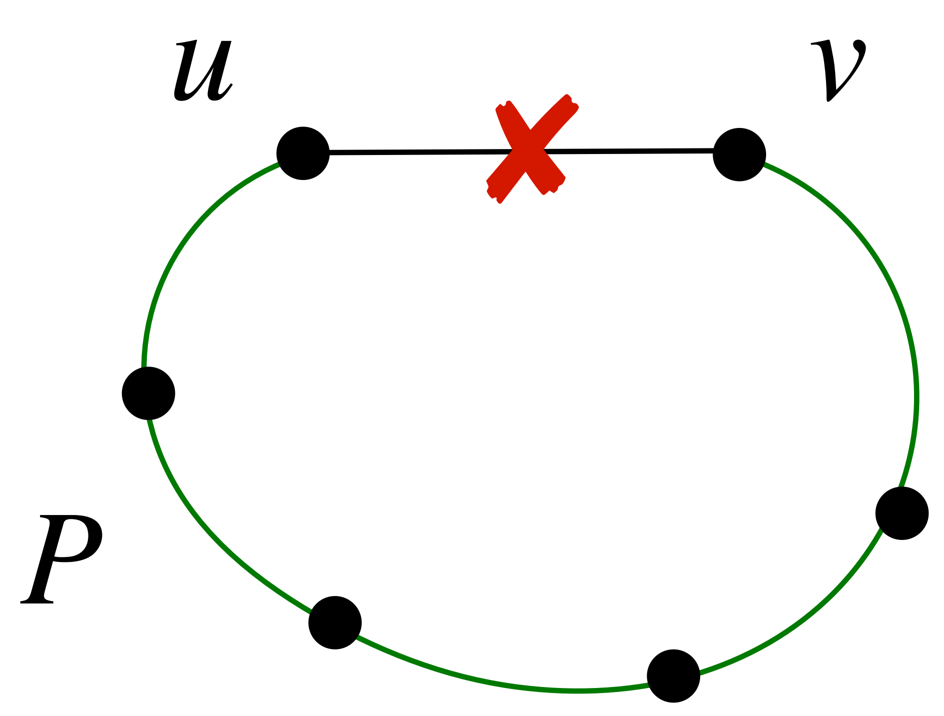
Alternative Characterisation of Trees

$2) \implies 3)$: Take any edge $e = \{u, v\}$ in G.



If $G - e$ is still connected, then there must be a path from u to v, say P.

But this implies that there are two paths from u to v in G, i.e., P and $\langle u, v \rangle$.

Hence, a contradiction.

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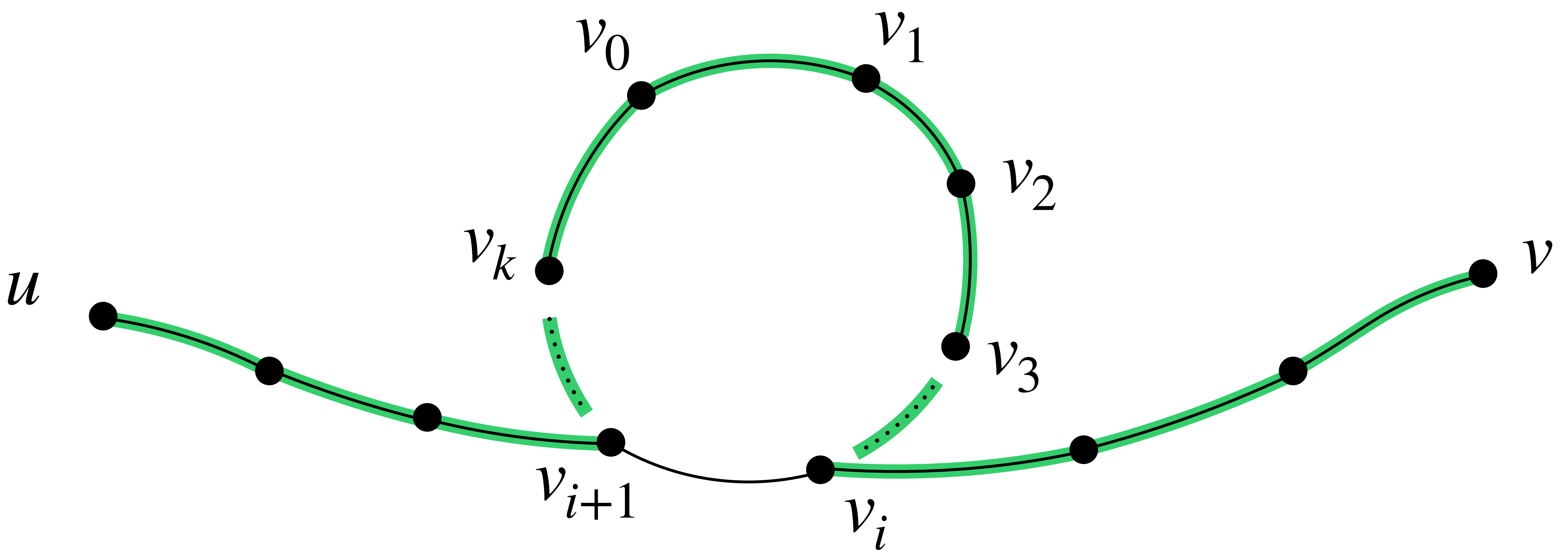
Alternative Characterisation of Trees

$3) \implies 1)$: If G is minimally connected, then G is connected.

Suppose G contains a cycle C. Let $e = \{v_i, v_{i+1}\}$ be an edge in C.

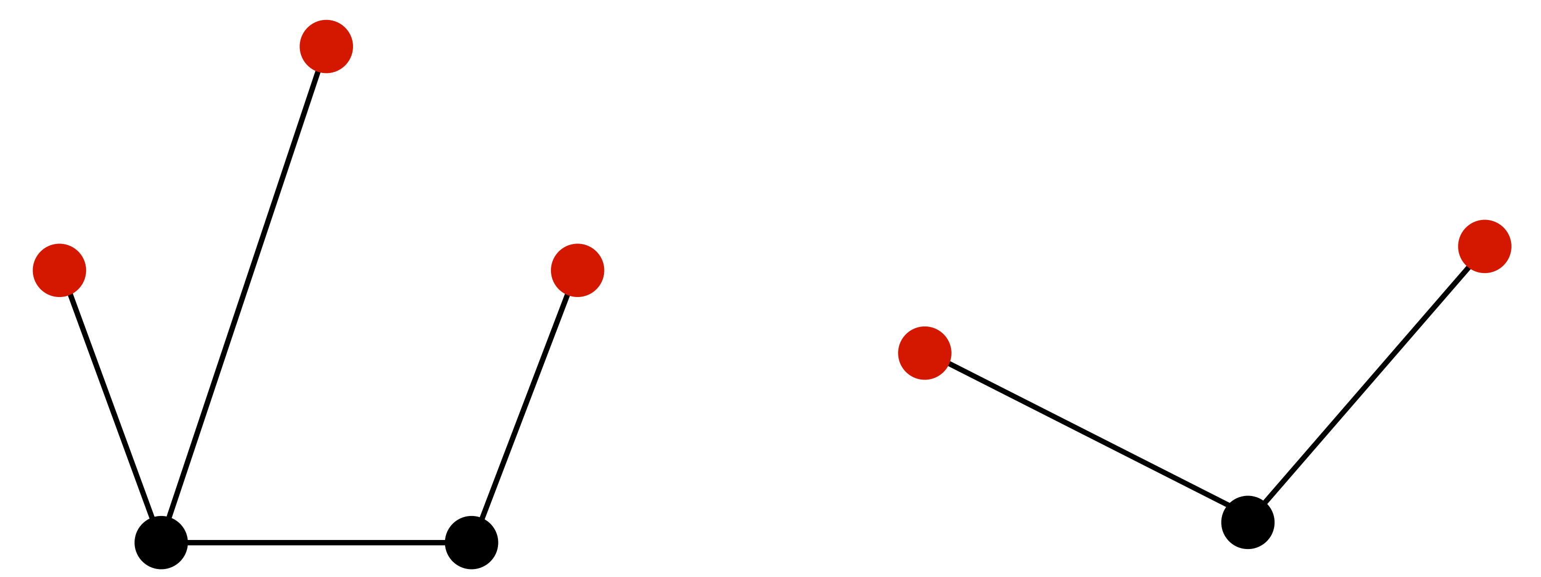


We now show that $G - e$ is still connected, hence a contradiction. Let $u, v \in G$.

- If the path P from u to v in G doesn't use e, then P will exist in $G - e$.
- If the path from u to v in G uses e, then we can replace e in P with $C - e$ to create a "walk" from u to v. And a "walk" from u to v, implies a path from u to v. ∎

Leaves in a Tree

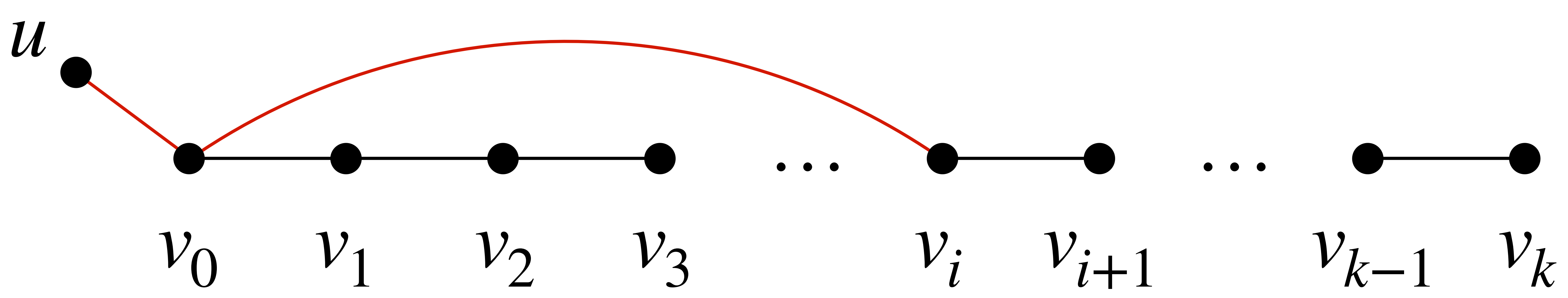
Definition: **Leaves** in a tree are vertices of degree one.



Theorem: Every tree of at least two vertices contains at least two leaves.

Proof: Let $P = \langle v_0, v_1, v_2, \ldots, v_k \rangle$ be a longest path in a tree T.

$P =$



v_0 cannot have a neighbour not in P. Otherwise, we can extend P to a longer path.

v_0 cannot have a neighbour apart from v_1 in P. Otherwise, it will create a cycle.

Hence, degree of v_0 is one. Similarly, v_k is also of degree one. ∎

Edges in a Tree

Theorem: Every tree of n vertices contains $n - 1$ edges.

Proof: We will prove it using induction.

Basis Step: Trivially true for tree of 1 vertex.

Inductive Step: Assume the theorem is true for all the trees of k many vertices and prove it for tree of $k + 1$ vertices.

Let T be a tree of $k + 1$ vertex and $T' = T - u$, where u is a leaf.

Clearly, T' is a tree of k vertices.

From inductive hypothesis, T' contains $k - 1$ edges.

Deletion of u caused deletion of only one edge in T.

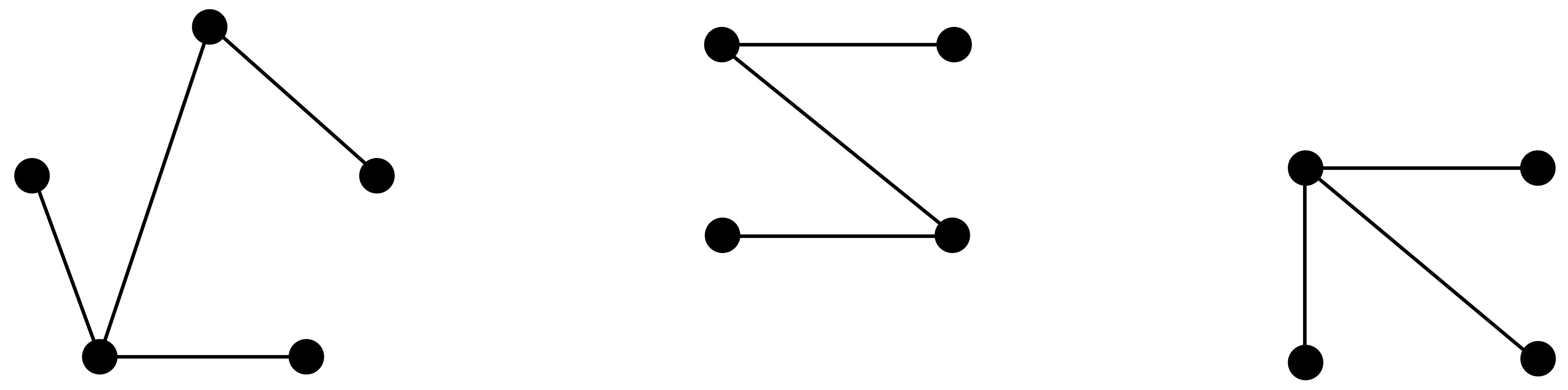
Hence, T contains exactly one more edge than T', i.e, k.

∎

Forest

Definition: A collection of trees is called a **Forest.**

Example: A forest of 13 vertices and 3 trees.



Theorem: Let F be a forest of n vertices and k trees. Then F contains $n - k$ edges.

Proof: DIY.